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The issue of the transformations of units is treated, mainly, in a geometrical context. Spacetime singularities are shown to be a consequence of a wrong choice of the geometrical formulation of the laws of gravitation. This result is discussed, in particular, for Friedmann-Robertson-Walker cosmology. It is also shown that Weyl geometry is a consistent framework for the formulation of the gravitational laws since the basic laws on which this geometry rests are invariant under the one-parameter Abelian group of units transformations studied in the paper. Riemann geometry does not fulfill this requirement. Arguments are given that point at Weyl geometry as a geometry implicitly containing the quantum effects of matter. The notion of geometrical relativity is presented. This notion may represent a natural extension of general relativity to include invariance under the group of units transformations.

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I. INTRODUCTION

Seemingly Dicke was the first physicist who called attention upon the importance of the transformations of units in physics [1]. Under a units transformation the coordinate system is held fixed. Hence, the labeling of the spacetime coincidences is invariant, while the curvature scalar and other purely geometrical scalars, invariant under diffeomorphisms, are generally not invariant under a units transformation. Therefor, spacetime measurements (observations) being nothing more than verifications of the spacetime coincidences, are invariant too under general transformations of units. Moreover, it is evident that the particular values of the units of measure employed are arbitrary, i.e., the physical laws must be invariant under a transformation of units [1]. This simple argument suggests that Einstein's general relativity(GR) in its actual form due to the action $S_{GR} = \int d^4x\sqrt{-g}(R + 16\pi L_{matter})$, where R is the curvature scalar and L_{matter} is the Lagrangian for the matter fields, is not a consistently formulated theory of spacetime. In fact, since the scalar R changes under a general transformation of units, then the laws of gravitation derived from the canonical action for GR change too. This conclusion is not a new one. Einstein's GR is intrinsically linked with the occurrence of spacetime singularities and, it is the hope that, when quantum effects would be included, the singularities would be removed from the description of the physical world. Other arguments showing that canonical Einstein's GR is not a consistently formulated theory of gravitation, come from string theory. This theory suggests that a scalar field (the dilaton) should be coupled to gravity in the low-energy limit of the theory [2].

Brans-Dicke(BD) theory of gravitation (and scalar-tensor(ST) theories in general) represents a natural generalization of Einstein's GR. This theory was first presented in reference [3] and subsequently reformulated by Dicke [1] in a conformal frame where the BD gravitational laws seemed like the Einstein's laws of gravitation. Both formulations of BD theory are linked by the conformal rescaling of the spacetime metric,

$$\hat{g}_{ab} = \Omega^2 g_{ab}, \tag{1.1}$$

where Ω^2 is a smooth, nonvanishing function on the spacetime manifold. Under (1.1) the coordinate system is held fixed as we have already remarked. This transformation can be viewed as a particular transformation of units: a point-dependent scale factor applied to the units of length, time and reciprocal mass [1]. The original formulation of BD theory was based on the following action [3]:

$$S_{BD} = \int d^4x\sqrt{-\hat{g}}(\hat{\psi}\hat{R} - \frac{\omega}{\hat{\psi}}(\hat{\nabla}\hat{\psi})^2 + 16\pi L_{matter}), \tag{1.2}$$

where ω is the BD coupling constant (a free parameter of the theory). Under (1.1) with $\Omega^2 = \hat{\psi}^{-1}$, (1.2) is mapped into the conformal formulation of BD theory [1],

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$$S_{BD} = \int d^4x\sqrt{-g}(R - (\omega + \frac{3}{2})(\nabla\psi)^2 + 16\pi e^{-2\psi}L_{matter}), \tag{1.3}$$

where the scalar field has been redefined ($\hat{\psi} \to e^{\psi}$). Apparently, Dicke did not note that the gravitational laws derived from (1.2) change under (1.1) with $\Omega^2 = \hat{\psi}^{-1}$ that he claimed to be a transformation of units. It is evident from the different forms of the actions (1.2) and (1.3).

Yet another important question is linked with the transformations of units of the kind (1.1). Under this transformation the laws of Riemann geometry change and, what appears to be a Riemann manifold with metric $\mathbf{g}$, is transformed under (1.1) into a manifold on Weyl geometry with metric $\hat{\mathbf{g}}$ [4,5]. The fact that the arbitrariness in the metric tensor (due to the arbitrariness in the choice of the units of measure) raises questions about the significance of Riemann geometry in relativity, was advanced by Brans and Dicke in reference [3]. In that paper the authors made evident their hope that the physical content of the theory should be contained in the invariants of the group of position-dependent transformations of units and coordinate transformations [3]. The last part of this hope (program) has been already completed in the seventies (see for instance [6]). However, I feel, the first part has not been worked out sufficiently.

The present paper has been motivated, in part, by the ambitious program Brans and Dicke outlined in [3] and, in part, by the long standing confusion the interpretation of transformation (1.1) has brought into gravitational physics. We do not pretend to resolve these profound questions in this paper. Our aim here is of more limited scope. Our goal is to present a point of view about the particular transformations of units (1.1) and the consequences it leads to the description of the physical world. We hope, this will be the beginning of a necessary and long avoided discussion in physics. Although we base our discussion mainly on geometrical considerations, the consequences of our reasoning line for gravitational theories are analysised in detail.

The paper has been organized in the following way. In section **II** Weyl geometry is shown to be a geometry conformal to Riemann one under (1.1). The consequences of the linkage of Weyl geometry with an effective theory of spacetime are outlined. The issue of the spacetime singularities will be treated in section **III**. It will be shown that singularities that occur in Riemannian spacetimes are removed (not apparently but effectively) in their conformal Weyl spacetimes that is made evident, as illustration, for Friedmann-Robertson-Walker cosmology. Section **IV** is the main part of this paper. It is fully devoted to the subject of units transformations of the kind (1.1). It will be shown, in particular, that the conformal transformation used to go from one formulation of BD and GR theories into their conformal formulations (transformation (1.1) with $\Omega^2 = \hat{\psi}^{-1}$) is not properly a units transformation. In this section we show also, that Weyl geometry is a consistent framework where to formulate the gravitational laws. In fact, the basic geometrical laws on which Weyl geometry rests are invariant in respect to the one-parameter Abelian group of units transformations we study here. Riemann geometry does not share this invariance. Hence, the conformal formulation of Einstein's GR, being naturally linked with Weyl geometry, gives a consistent formulation of the laws of gravity unlike BD theory and the canonical formulation of GR. In section **V** we present some considerations in favour of Weyl geometry as an intrinsically quantum geometry. Part of these considerations are based in the de Broglie-Bohm quantum theory of motion [7] and on an idea advanced in reference [8]. Finally, in section **VI** we discuss on the meaning of the conformal transformation (1.1) for geometry. We are led to the notion of geometrical relativity that, we hope, will represent a natural extension of general relativity to include invariance under the group of units trnasformations.

II. WEYL GEOMETRY

Usually gravity theories such like Einstein's general relativity and Brans-Dicke theory (and scalar-tensor theories in general) are linked with Riemann geometry.

Suppose $\mathbf{X}$, $\mathbf{Y}$ are C^1 vector fields and $\lambda(t)$ is a curve on the manifold. If $\mathbf{X}$ is the tangent vector to $\lambda(t)$ and choosing local coordinates so that λ has the coordinates x^a, i.e. $X^a = \frac{dx^a}{dt}$, then the covariant derivative of $\mathbf{Y}$ along λ can be written as [6]

$$\frac{DY^a}{\partial t} = \frac{\partial Y^a}{\partial t} + \gamma^a_{mn} Y^m \frac{dx^n}{dt}, \tag{2.1}$$

where γ^a_{bc} are the components of the affine connections. The vector $\mathbf{Y}$ is said to be parallely transported along λ if $\frac{D\mathbf{Y}}{\partial t} = 0$. The curve $\lambda(t)$ is said to be a geodesic curve if $\frac{D\mathbf{X}}{\partial t}$ is parallel to $\mathbf{X}$. In terms of an affine parameter v along λ the associated tangent vector $\mathbf{V} = (\frac{\partial}{\partial v})_\lambda$ is parallel to $\mathbf{X}$ but has its 'length' defined through $V(v) = 1$. It obeys the equations $V^a_{;n} V^n = 0$. In particular, if $\mathbf{X}$ is a time-like vector the affine parameter v can be set equal to the arc length s. Hence the local coordinate expression for the geodesic equation can be written as

$$\frac{d^2x^a}{ds^2}+\gamma^a_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds}=0. \tag{2.2}$$

Given a metric $\mathbf{g}$ on the manifold $\mathcal{M}$, the Riemann geometry is fixed by the following postulate. There is a unique torsion-free connection on $\mathcal{M}$ defined by the condition that the covariant derivative of $\mathbf{g}$ is zero, i.e. $g_{ab;c}=0$. With the connection defined in such a way, parallel transfer of vectors preserves scalar products defined by $\mathbf{g}$. In particular

$$dg(\mathbf{Y},\mathbf{Y})=0, \tag{2.3}$$

where $g(\mathbf{Y},\mathbf{Y})=g_{mn}Y^mY^n$. Eq.(2.3) together with the parallel transfer law $\frac{D\mathbf{Y}}{\partial t}=0$ lead that the affine connections on a Riemann manifold coincide with the Christoffel symbols of the metric $\mathbf{g}$

$$\gamma^a_{bc}=\Gamma^a_{bc}=\frac{1}{2}g^{an}(g_{bn,c}+g_{cn,b}-g_{bc,n}). \tag{2.4}$$

The postulate (2.3) is realized in general relativity and BD theory [3] through the assumption that there exists physical systems such like atoms that have physical properties independent of location [3]. This is equivalent to say that one can take some quantities associated with these systems (for instance the atom radius and transition energies) as one's units of measurement. These will be constant over the manifold. Hence, for instance, the arc-length between two sucessive events on a geodesic curve will be poit-independent as required by (2.3). However, as noted by Dicke [1] there may be more than one feasible way of stablishing the equality of units at different spacetime points. Such as it is necessary first to make a choice of the unit of length before a spacetime geometry is established [1] hence, there may be more than one feasible spacetime geometry that can be taken to model our real world. I think a similar argument led Brans and Dicke to raise questions about the significance of Riemann geometry in relativity [3]. In the remainder of this section we shall illustrate this fact. A conclusive discussion on this will be given in section **VI**.

We shall study the consequences of the conformal rescaling of the metric $\mathbf{g}\rightarrow\Omega^{-2}\hat{\mathbf{g}}$(eq.(1.1)) for the spacetime geometry. As remarked in section **I** it is the mathematical representation of a particular transformation of units (a position-dependent scale factor applied to the units of length, time and reciprocal mass [1]). Since the physical laws must be invariant under these transformations then, the corresponding conformal geometries must be physically equivalent.

Under (1.1) the covariant derivative of the vector $\mathbf{Y}$(eq.(2.1)) with $\gamma^a_{bc}=\Gamma^a_{bc}$ is mapped into

$$\frac{DY^a}{\partial t}=\frac{\partial Y^a}{\partial t}+\hat{\Gamma}^a_{mn}Y^m\frac{dx^n}{dt}-\Omega^{-1}(\Omega_{,n}Y^n\frac{dx^a}{dt}+\Omega_{,n}\frac{dx^n}{dt}Y^a-\hat{g}_{mn}Y^m\frac{dx^n}{dt}\hat{g}^{as}\Omega_{,s}), \tag{2.5}$$

where $\hat{\Gamma}^a_{bc}$ are the Christoffel symbols of the metric with hat. New affine connections can be defined

$$\hat{\gamma}^a_{bc}=\hat{\Gamma}^a_{mn}-\Omega^{-1}(\Omega_{,c}\delta^a_b+\Omega_{,b}\delta^a_c-\hat{g}_{bc}\hat{g}^{as}\Omega_{,s}), \tag{2.6}$$

such that (2.5) takes a form similar to (2.1)

$$\frac{DY^a}{\partial t}=\frac{\partial Y^a}{\partial t}+\hat{\gamma}^a_{mn}Y^m\frac{dx^n}{dt}. \tag{2.7}$$

The affine connections with hat are not connections on a Riemann manifold since they do not coincide with the Christoffel symbols of the corresponding metric. In fact, under (1.1) the postulate (2.3) of length preservation in Riemann geometry changes into the following postulate of length transport

$$d\hat{g}(\mathbf{Y},\mathbf{Y})=2\Omega^{-1}dx^n\Omega_{,n}\hat{g}(\mathbf{Y},\mathbf{Y}), \tag{2.8}$$

where now the scalar product $\hat{g}(\mathbf{Y},\mathbf{Y})$ is given in terms of $\hat{\mathbf{g}}$. The new geometry based on the postulate (2.8) and the parallel transport law $\frac{DY^a}{\partial t}=0$ with the covariant derivative defined through (2.7), is a particular case of the well-known Weyl geometry. For a time-like tangent vector $\hat{\mathbf{X}}$ with coordinates $\frac{dx^a}{d\hat{s}}$ the geodesic equation (2.2) can be written as

$$\frac{d^2x^a}{d\hat{s}^2}+\hat{\gamma}^a_{mn}\frac{dx^m}{d\hat{s}}\frac{dx^n}{d\hat{s}}+\Omega^{-1}\Omega_{,n}\frac{dx^n}{d\hat{s}}\frac{dx^a}{d\hat{s}}=0, \tag{2.9}$$

where we have considered that $ds=\Omega^{-1}d\hat{s}$. Eq.(2.9) can be written more explicitly as

$$\frac{d^2x^a}{d\hat{s}^2}+\hat{\Gamma}^a_{mn}\frac{dx^m}{d\hat{s}}\frac{dx^n}{d\hat{s}}-\Omega^{-1}\Omega_{,n}(\frac{dx^n}{d\hat{s}}\frac{dx^a}{d\hat{s}}-\hat{g}^{na})=0. \tag{2.10}$$

These equations (eq.(2.9) or (2.10)) define a time-like geodesic curve in a Weyl manifold. The units of the Weyl geometry change length from point to point in spacetime according to the law (2.8). Hence this geometry represents a generalization of Riemann geometry. When a physical theory of spacetime is incorporated, some conclusions raise. First, Weyl geometry can not be linked neither with the Jordan frame formulation of BD theory [3] nor with the usual Einstein's formulation of GR since, their implicit postulate is that physical units of measure (realized through physical systems like atoms) are constant over spacetime. Weyl geometry is naturally linked, in particular, with the Einstein frame formulation of Brans-Dicke gravity [1] and with the Jordan frame formulation of general relativity [4,5,9]. The second conclusion is linked with the fact that the formulation of a given spacetime theory compatible with Weyl geometry should contain a second order differential equation for determining the scalar function Ω. Hence, particular solutions of this differential equation produce particular functional laws of change of the units of measure on the manifold and hence, different Weyl geometries. I. e., Weyl geometry is a whole class of physically equivalent geometries. However, some of these geometries must be dropped if the corresponding solutions of the differential equation for Ω can be dropped on the basis of physical considerations. This class contains the Riemann geometry as a particular member if $\Omega = const.$ is a solution of this differential equation.

A third consequence is yet allowed by the incorporation of a physical theory of spacetime compatible with Weyl geometry. It is related with the spacetime singularities that usually arise in Riemannian manifolds. In fact, the theorems on singularities do not depend on the full equations of gravity but only on the property that $R_{mn}K^mK^n$ is non-negative for any non-spacelike vector K^a. Hence they would apply as well to any modification of general relativity (such as BD theory) in which gravity is always attractive [6]. Under (1.1) the Ricci tensors with a hat and without it are related through

$$R_{ab} = \hat{R}_{ab} - 3\hat{g}_{ab}\Omega^{-2}(\hat{\nabla}\Omega)^2 + \Omega^{-1}(2\Omega_{;ab} + \hat{g}_{ab}\Box\Omega), \tag{2.11}$$

where the covariant derivative in the right hand side(rhs) of eq.(2.11) is given in respect to the metric $\hat{\mathbf{g}}$. Hence the condition $R_{mn}K^mK^n \geq 0$ is transformed into the following condition:

$$\hat{R}_{mn}\hat{K}^m\hat{K}^n - 3\hat{g}_{mn}\hat{K}^m\hat{K}^n\Omega^{-2}(\hat{\nabla}\Omega)^2 + 2\Omega^{-1}\Omega_{;mn}\hat{K}^m\hat{K}^n + \hat{g}_{mn}\hat{K}^m\hat{K}^n\Omega^{-1}\Box\Omega) \geq 0, \tag{2.12}$$

where the non-spacelike vector $\hat{K}^a(\hat{g}_{mn}\hat{K}^m\hat{K}^n \leq 0)$ is related with K^a through $\hat{K}^a = \Omega^{-1}K^a$. This means that the conditon (2.12) may be fulfilled even if $\hat{R}_{mn}\hat{K}^m\hat{K}^n < 0$, and correspondingly some singularity theorems may not hold in the conformal frame. Hence, spacetime singularities occurring in Riemann geometry may be removed in some of its equivalent Weyl geometries generated by the physically meaningless transformation of units (1.1). This will be the subject of the following section.

Finally we shall remark that, up to this moment in our discussion, neither Riemann geometry nor any of its conformal Weyl geometries is preferred on physical grounds since, they are related through the particular units transformation (1.1) that preserves unchanged the physical laws. Moreover, experimental observations being nothing more than verifications of the spacetime coincidences, are unchanged under (1.1). Recall that the labeling of the spacetime coincidence between two particles is invariant by definition in respect to these transformations [1]. This means that, even experimental measurements are unable to differentiate these geometries. Nevertheless, as we shall see below, this 'duality' of the geometrical interpretation of the laws of gravity [4], can be effectively removed.

III. WEYL GEOMETRY AND SPACETIME SINGULARITIES

Without loss of generality we shall study the Raychaudhuri equation for a congruence of time-like geodesics without vorticity, with the time-like tangent vector V^a [6]:

$$\frac{d\Theta}{ds} = -R_{mn}V^mV^n - 2\sigma^2 - \frac{1}{3}\Theta^2, \tag{3.1}$$

where Θ is the volume expansion of the time-like geodesic and σ is the shear. As seen from (3.1) Θ will monotonically decrease along the time-like geodesic if $R_{mn}K^mK^n \geq 0$ for any time-like vector $\mathbf{K}$(the time-like convergence condition).

Under (1.1) the Raychaudhuri equation (3.1) is mapped into:

$$\frac{d\hat{\Theta}}{d\hat{s}} = -\hat{R}_{mn}\hat{V}^m\hat{V}^n - 2\hat{\sigma}^2 - \frac{1}{3}\hat{\Theta}^2 + \Omega^{-1}\Omega_{,n}\hat{V}^n\hat{\Theta} + \Omega^{-1}(\Omega_{;nm} - 3\frac{\Omega_{,n}\Omega_{,m}}{\Omega})\hat{h}^{nm}, \tag{3.2}$$

where $\hat{h}^{ab} = \hat{g}^{ab} + \hat{V}^a\hat{V}^b$. The additional fourth and fifth terms in the rhs of eq.(3.2) have non-definite sign and hence can, in principle, contribute to expansion instead of contraction. When the contribution to expansion of these terms (if they effectively contribute to expansion) becomes stronger than the contribution to focusing of the first three terms, then contraction changes into expansion and no singularity occurs. In this case wormhole spacetimes are allowed instead of singular ones. A less ambiguous discussion of this subject could be given only after incorporation of an effective theory of spacetime.

A. General relativity with an extra scalar field

As an effective theory of spacetime we shall approach general relativity with an extra scalar field ψ, given by the canonical action

$$S_{GR} = \int d^4x\sqrt{-g}(R - \alpha(\nabla\psi)^2 + 16\pi L_{matter}), \tag{3.3}$$

where $\alpha(\alpha \geq 0)$ is a free parameter and L_{matter} is the Lagrangian for the ordinary matter fields. The equations derivable from (3.3) are

$$G_{ab} = 8\pi T_{ab} + \alpha(\psi_{,a}\psi_{,b} - \frac{1}{2}g_{ab}(\nabla\psi)^2), \tag{3.4}$$

where the gravitational constant $G = 1$, $G_{ab} \equiv R_{ab} - \frac{1}{2}g_{ab}R$, $T_{ab} = \frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}L_{matter})}{\partial g^{ab}}$ is the stress-energy tensor for the matter fields, and $\frac{1}{8\pi}$ times the 2nd term in the rhs of eq.(3.4) is the stress-energy tensor for the scalar field. The following wave equation for ψ is also derivable from the action (3.3),

$$\Box\psi = 0. \tag{3.5}$$

The stress-energy tensor T_{ab} fulfills the conservation equation

$$T^{an}_{;n} = 0. \tag{3.6}$$

For illustration we shall study homogeneous and isotropic universes with the Friedmann-Robertson-Walker(FRW) line-element (we use coordinates t, r, ϑ, φ)

$$ds^2 = -dt^2 + a^2(t)(\frac{dr^2}{1-kr^2} + r^2d\Omega^2), \tag{3.7}$$

where $a(t)$ is the scale factor, $d\Omega^2 = d\vartheta^2 + \sin^2\vartheta d\varphi^2$ and $k = 0$ for flat, $k = -1$ for open and $k = +1$ for closed universes. The universe is supposed to be filled with a barotropic perfect fluid with the barotropic equation of state $p = (\gamma - 1)\mu$, where μ is the energy density of matter and the barotropic index $0 < \gamma < 2$. The perfect fluid stress-energy tensor is $T_{ab} = (\mu + p)V_aV_b + pg_{ab}$. Equations (3.4), (3.5) and (3.6) lead to the following eqution for determining the scale factor

$$(\frac{\dot{a}}{a})^2 + \frac{k}{a^2} = \frac{M}{a^{3\gamma}} + \frac{\alpha A^2}{a^6}, \tag{3.8}$$

where the overdot means derivative with respect to the proper time t. M and A are arbitrary integration constants. While deriving eq.(3.8) we have considered that after integrating eq.(3.5) once we obtain $\dot{\psi} = \pm\frac{\sqrt{6}A}{a^3}$, while (3.6) gives $\mu = \frac{3M}{8\pi a^{3\gamma}}$.

If we take the time-like tangent vector of a comoving observer $V^a = \delta^a_0$, then the Raychaudhuri equation (3.1) can be written as

$$\dot{\Theta} = -\frac{9}{2}\frac{\gamma M}{a^{3\gamma}} - \frac{9\alpha A^2}{a^6} + \frac{3k}{a^2}, \tag{3.9}$$

where we took the reversed sense of the proper time $-\infty \leq t \leq 0$, i.e., $a(t)$ runs from infinity to zero. The well-known results of GR emerge from a careful analysis of this equation. For $k = 0$ and $k = -1$ eq.(3.9) gives focusing of the fluid lines, leading to a global singularity at $t = 0$ where the density of matter becomes infinite and the known physical laws breakdown. For $k = +1$ $a(t)$ is a monotonic function of the proper time. It grows from zero to a maximum value and then decreases to zero. Eq.(3.9) gives focusing of the fluid lines twice (the value $a = 0$ occurs twice) as required. The closed universe emerges from a global initial singularity, grows and then merges into a final singularity. All these results are well-known in general relativity on Riemann manifolds. Our goal here is to show what happens when we formulate general relativity on a Weyl manifold.

If we set Ω in eq.(1.1) to be $\Omega^2 = e^{-\psi}$ then the Raychaudhuri equation in the corresponding Weyl manifold can be written in terms of the Riemann scale factor $a(t)$ as

$$\frac{d\hat{\Theta}^{\pm}}{d\tau} = e^{\psi^{\pm}}\left(-\frac{9}{2}\frac{\gamma M}{a^{3\gamma}} - \frac{9(\alpha+\frac{1}{2})A^2}{a^6} + \frac{3k}{a^2} \pm \frac{6\sqrt{6}A}{a^3}\sqrt{\frac{\gamma M}{a^{3\gamma}} + \frac{\alpha A^2}{a^6} - \frac{k}{a^2}}\right), \tag{3.10}$$

where $\tau = \int e^{-\frac{\psi}{2}} dt$ is the proper time in the Weyl spacetime generated by the transformation (1.1) with $\Omega = e^{-\frac{\psi}{2}}$ applied to the FRW (Riemann) spacetime with the line-element (3.7). For simplicity we shall concentrate on flat ($k = 0$) and open ($k = -1$) universes (the case $k = +1$ requires a very detailed analysis). For big a eq.(3.10) shows that focusing of the fluid lines occurs. As we go backwards in time t, a decreases and, for sufficiently small $a \ll 1$, eq.(3.10) can be written as

$$\frac{d\hat{\Theta}^{\pm}}{d\tau} \approx \frac{3A^2 e^{\psi^{\pm}}}{a^6}\left(-3\alpha \pm 2\sqrt{6}\sqrt{\alpha} - \frac{3}{2}\right). \tag{3.11}$$

If we choose the '+' branch of the solution of the wave equation (3.5)

$$\psi^{\pm} = \psi_0 \pm \sqrt{6A}\int \frac{da}{a^3\dot{a}}, \tag{3.12}$$

that is given by the choice of the '+' sign in (3.12) (ψ_0 is an integration constant), then when α is in the range $0 \leq \alpha \leq \frac{1}{6}$ and for small enough a, eq.(3.10) (and correspondingly eq.(3.11)) shows that contraction of fluid lines turns into expansion and no singularity is formed in the Weyl spacetime conformal to the Riemannian FRW one, that is given by the line-element (3.7). In fact, while the curvature scalar R for the FRW spacetime linked with the barotropic fluid,

$$R = \frac{3}{a^6}[(4-3\gamma)Ma^{3(2-\gamma)} - 2\alpha A^2], \tag{3.13}$$

is singular at $a = 0$, its conformal (in the '+' branch and for small a) behaves like

$$\hat{R}^{+} \sim -3(2\alpha - 3)A^2 e^{\phi_0} a^{\sqrt{\frac{6}{\alpha}}-6}. \tag{3.14}$$

For $\alpha \leq \frac{1}{6}$, $\hat{R}^{+}$ is bounded even for $a = 0$. Moreover, for $k = 0$ and $k = -1$, and $0 \leq \alpha \leq \frac{1}{6}$, $\hat{R}^{+}$ is regular and bounded everywhere in the range $0 \leq a \leq \infty$. In the Weyl spacetime the fluid lines converge into the past up to the moment when a becomes small enough and then they diverge. This is the way the spacetime singularity that always occurs in the FRW (Riemannian) spacetime given by (3.7), is removed in its conformal Weyl spacetime.

The following question is to be raised. Is illusory the vanishing of the cosmological singularity in the Weyl spacetime?. In other words: would test particles feel a singularity in the Weyl geometry even if the geometry seems to be that of a cosmological wormhole?. In this sense we shall remark that the inevitabillity of the cosmological singularity in the Riemann manifold with the FRW line-element (3.7) is linked with the fact that the geodesic lines are incomplete into the past (the proper time t is constrained to the range $0 \leq t \leq +\infty$). In its conformal wormhole spacetime given by the line-element

$$d\hat{s}^2 = -d\tau^2 + \hat{a}^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right), \tag{3.15}$$

where $\hat{a} = e^{-\frac{\psi}{2}}a$, the geodesics (see eq.(2.10) for timelike geodesics in a Weyl manifold) are complete in the Weyl manifold. In particular, when t goes over the range $0 \leq t \leq +\infty$, τ goes over the range $-\infty \leq \tau \leq +\infty$. While the matter density seen by a comoving observer in the Riemann geometry $T_{mn}V^mV^n = \mu$ ($V^a = \delta^a_0$) is singular at $t = 0 (a = 0)$, the corresponding energy density as measured by a comoving observer in Weyl geometry ($\hat{V}^a = e^{\frac{\psi^+}{2}}V^a$), $\hat{T}_{mn}\hat{V}^m\hat{V}^n = e^{2\psi^+}\mu$ is regular and bounded for all times $0 \leq t \leq +\infty$ ($-\infty \leq \tau \leq +\infty$). The same is true for the energy density of the scalar field μ_ψ measured by a comoving observer. Hence, the absence of spacetime singularities in a Weyl spacetime (conformal to a singular Riemann spacetime) is a real feature of this geometry that can be tested with the help of test particles.

We should explain yet another thing. The vanishing of the cosmological singularity in the Weyl spacetime is allowed only in the '+' branch of the solution to the wave equation for ψ. In the '-' branch the Weyl spacetime is singular too (like the Riemann one). Hence we should give a physical consideration why we chose the '+' branch. In this sense we shall note that under the conformal transformation (1.1) with $\Omega^2 = e^{-\psi}$, the action (3.3) maps into its conformal

$$S_{GR} = \int d^4x\sqrt{-\hat{g}}e^{\psi}(\hat{R} - (\alpha - \frac{3}{2})(\hat{\nabla}\psi)^2 + 16\pi e^{\psi}L_{matter}). \tag{3.16}$$

Hence in the Weyl manifold $e^{-\psi}$ plays the role of an effective gravitational constant $\hat{G}$. For the '-' branch $\hat{G}$ runs from zero to an infinite value, i.e., gravity becomes stronger with the evolution of the universe and in the infinite future it dominates over the other interactions (or becomes of the same range), that is in contradiction with the usual picture. On the contrary, for the '+' branch, $\hat{G}$ runs from an infinite value to zero as the universe evolves and, hence, gravitational effects are weakened as required. The fact that, in this branch, the vanishing of the singularity is effective only for $0 \leq \alpha \leq \frac{1}{6}$ can be taken only as a restriction on the values the free parameter α can take.

Exact analytic solutions for Jordan frame general relativity with a barotropic perfect fluid can be found in reference [5] for flat FRW cosmology and in reference [9] for open dust-filled and radiation-filled universes.

IV. WEYL GEOMETRY AND TRANSFORMATIONS OF UNITS

As we pointed out in section **I**, Dicke was first physicist who noticed the importance of the transformations of units in gravitation theory [1]. He studied, in particular, a units transformation of the kind (1.1), i.e., a point-dependent scale factor applied to the units of length, time and reciprocal mass. Dicke used the transformation (1.1) with $\Omega^2 = e^{-\psi}$ to formulate the BD theory, given by the string frame action

$$S_{BD} = \int d^4x\sqrt{-\hat{g}}e^{\psi}(\hat{R} - \omega(\hat{\nabla}\psi)^2 + 16\pi e^{-\psi}L_{matter}), \tag{4.1}$$

(the change of variable $\hat{\psi} = e^{\psi}$ maps the action (4.1) into the well-known action (1.2) for BD theory in the Jordan frame), in the Einstein frame

$$S_{BD} = \int d^4x\sqrt{-g}(R - (\omega + \frac{3}{2})(\nabla\psi)^2 + 16\pi e^{-2\psi}L_{matter}). \tag{4.2}$$

Hence, although in a very subtle manner, Dicke recognized that either Brans-Dicke theory is not invariant under the units transformation he studied, or else transformation (1.1) with $\Omega^2 = e^{-\psi}$ is not properly a units transformation. In fact, it is evident that the particular values of the units of measure one employs are arbitrary so the physical laws should be invariant under general transformations of units.

In this section we shall extend these arguments to our geometrical discussion since, it is evident that the particular values of the units of the geometry should not influence the geometrical laws. Any consistent geometrical description should be insensible to the units one chooses. For the purposes of our discussion we shall take the following units transformation

$$\bar{g}_{ab} = e^{-\sigma\psi}g_{ab}, \tag{4.3}$$

and the scalar function redefinition

$$\bar{\psi} = (1 - \sigma)\psi, \tag{4.4}$$

where σ is some constant parameter ($\sigma \neq 1$). This transformation was introduced in reference [10] with a different definition of the scale factor and scalar field variable. It constitutes a one-parameter Abelian group. A composition of two successive transformations with parameters $\sigma_1 \neq 1$ and $\sigma_2 \neq 1$ yields a transformation of the same kind with parameter $\sigma_3 = \sigma_1 + \sigma_2 - \sigma_1\sigma_2 \neq 1$, such that $\sigma_3(\sigma_1, \sigma_2) = \sigma_3(\sigma_2, \sigma_1)$, i.e., $\sigma_1 \times \sigma_2 = \sigma_2 \times \sigma_1$ and the group is commutative. The identity of this group is the transformation with $\sigma = 0$. The inverse is the transformation with $\bar{\sigma} = \frac{\sigma}{\sigma - 1}$. We see that for $\sigma = 1$ the inverse does not exist. Hence, the transformation (1.1) with $\Omega^2 = e^{-\psi}$ is not properly a units transformation since it has no inverse, i.e., it does not constitute a group[1].

It can be easily checked that Riemann geometry with the parallel transport law (see section **II**)

[1]Dimensional analysis is an elementary group theoretic technique [1].

$$\frac{DY^a}{\partial t} = \frac{\partial Y^a}{\partial t} + \Gamma^a_{mn} Y^m \frac{dx^n}{dt} = 0, \tag{4.5}$$

and the length preservation law

$$dg(\mathbf{Y}, \mathbf{Y}) = 0, \tag{4.6}$$

is not invariant under (4.3), (4.4), that can be interpreted geometrically as a transformation of the units of length of the geometry. In fact, under (4.3) and (4.4), the parallel transport law (4.5) is mapped into

$$\frac{\partial Y^a}{\partial t} + \bar{\Gamma}^a_{mn} Y^m \frac{dx^n}{dt} + \frac{\sigma}{2(1-\sigma)}(\bar{\psi}_{,n} \frac{dx^n}{dt} Y^a + \bar{\psi}_{,n} Y^n \frac{dx^a}{dt} - Y_n \frac{dx^n}{dt} \bar{g}^{as} \bar{\psi}_{,s}), \tag{4.7}$$

while (4.6) is mapped into

$$d\bar{g}(\mathbf{Y}, \mathbf{Y}) = -\frac{\sigma}{1-\sigma} dx^n \bar{\psi}_{,n} \bar{g}(\mathbf{Y}, \mathbf{Y}). \tag{4.8}$$

In the same way, under the conformal rescaling (1.1) with $\Omega^2 = e^{-\psi}$ (that, as we have already shown, is not properly a units transformation), Riemann geometry based on the laws (4.5) and (4.6), transforms into a Weyl geometry with the parallel transport law (see section **II**)

$$\frac{\partial Y^a}{\partial t} + \hat{\gamma}^a_{mn} Y^m \frac{dx^n}{dt} = 0, \tag{4.9}$$

and the length transport law

$$d\hat{g}(\mathbf{Y}, \mathbf{Y}) = -dx^n \psi_{,n} \hat{g}(\mathbf{Y}, \mathbf{Y}). \tag{4.10}$$

In equation (4.9) the affine connections $\hat{\gamma}^a_{bc}$ are related with the Christoffel symbols of the metric $\hat{\mathbf{g}}$ through eq.(2.6) with $\Omega = e^{-\frac{\psi}{2}}$,

$$\hat{\gamma}^a_{bc} = \hat{\Gamma}^a_{mn} + \frac{1}{2}(\psi_{,c}\delta^a_b + \psi_{,b}\delta^a_c - \hat{g}_{bc}\hat{g}^{as}\psi_{,s}). \tag{4.11}$$

In this geometry time-like test particles follow geodesics that are given by the following equation

$$\frac{d^2x^a}{d\hat{s}^2} + \hat{\Gamma}^a_{mn} \frac{dx^m}{d\hat{s}} \frac{dx^n}{d\hat{s}} + \frac{1}{2}\psi_{,n}(\frac{dx^n}{d\hat{s}} \frac{dx^a}{d\hat{s}} - \hat{g}^{na}) = 0. \tag{4.12}$$

We shall test this geometry in respect to the transformation of units (4.3) and the scalar field redefinition (4.4) (in equation (4.3) we change $g_{ab} \to \hat{g}_{ab}$). It can be verified that under (4.3) and (4.4) the laws of parallel transport (4.9) and length transport (4.10) are invariant in form

$$\frac{\partial Y^a}{\partial t} + \bar{\gamma}^a_{mn} Y^m \frac{dx^n}{dt} = 0, \tag{4.13}$$

$$d\bar{g}(\mathbf{Y}, \mathbf{Y}) = -dx^n \bar{\psi}_{,n} \bar{g}(\mathbf{Y}, \mathbf{Y}), \tag{4.14}$$

where (compare with relationship (4.11))

$$\bar{\gamma}^a_{bc} = \bar{\Gamma}^a_{mn} + \frac{1}{2}(\bar{\psi}_{,c}\delta^a_b + \bar{\psi}_{,b}\delta^a_c - \bar{g}_{bc}\bar{g}^{as}\bar{\psi}_{,s}). \tag{4.15}$$

In particular the equation defining time-like geodesics in the Weyl manifold (eq.(4.12)) is invariant in form too under the transformation of units (4.3) and the scalar field redefinition (4.4)

$$\frac{d^2x^a}{d\bar{s}^2} + \bar{\Gamma}^a_{mn} \frac{dx^m}{d\bar{s}} \frac{dx^n}{d\bar{s}} + \frac{1}{2}\bar{\psi}_{,n}(\frac{dx^n}{d\bar{s}} \frac{dx^a}{d\bar{s}} - \bar{g}^{na}) = 0. \tag{4.16}$$

After these results some conclusions raise. First, the conformal transformation studied in subsection **A** of section **III** ($\Omega^2 = e^{-\psi}$, i.e., $\sigma = -1$ in eq.(4.3)) and used in the literature to 'jump' from one formulation of scalar tensor gravity to its conformal formulation, does not constitute a group and hence, it can not be taken properly as a transformation of units. Unlike this, transformation (4.3) together with (4.4) ($\sigma \neq 1$) can be properly interpreted as a units transformation. Second, Weyl geometry is invariant under this particular transformation of units while Riemann geometry it is not. The linkage of these results leads us, inevitably, to the conclusion that Weyl geometry is a consistent framework for the interpretation of the physical world since, in physics, the transformation of units is meaningless. Riemann geometry, conformal to it, is not such a consistent framework.

What happens when we approach an effective theory of spacetime?. Take, for instance, the Brans-Dicke theory given by the string frame action (4.1). It can be verified that (4.1) is invariant in form under (4.3) and (4.4) only for pure BD or for BD with ordinary matter content with a trace-free stress-energy tensor [10][2]. On the other hand, Brans-Dicke theory is naturally linked with Riemann geometry that is not invariant under the units transformation (4.3) and the field redefinition (4.4). This means that, in its present form, BD theory can not be considered as a consistent theory of spacetime. The Einstein frame formulation of this theory is linked with Weyl geometry that is itself invariant under (4.3) and (4.4). However, the Einstein frame action for BD theory is not invariant in form under this units transformation even for pure gravity so, it is not a consistent formulation of the laws of gravity too. The same is true for the canonical Einstein's general relativity due to the action (3.3). Moreover, this formulation of general relativity is linked with Riemann geometry that, as we have just remarked, is not invariant in form under (4.3) and (4.4). This means that canonical Einstein's GR is not yet a consistent theory of spacetime. This conclusion is not a new one. It is well-known that canonical general relativity should be completed with quantum effects. It is the hope that, when such a quantum theory of gravity will be worked out, it should be invariant under units transformations of the kind (4.3), (4.4).

The situation changes when we approach the conformal formulation of general relativity given by the action (3.16) in the string frame or by

$$S_{GR} = \int d^4x\sqrt{-\hat{g}}(\hat{\psi}\hat{R} - (\alpha - \frac{3}{2})\frac{(\hat{\nabla}\hat{\psi})^2}{\hat{\psi}} + 16\pi\hat{\psi}^2 L_{matter}), \tag{4.17}$$

in the Jordan frame ($\hat{\psi} = e^{\psi}$). This action (and correspondingly action (3.16)) is invariant in form under (4.3) and (4.4), together with the parameter transformation

$$\bar{\alpha} = \frac{\alpha}{(1-\sigma)^2}. \tag{4.18}$$

For $\sigma = 2$ the transformation (4.3), (4.4) ($\bar{g}_{ab} = e^{-2\psi}\hat{g}_{ab}, \bar{\psi} = -\psi$) does not touch the free parameter of the theory α. The invariance of the pure gravitational part of (4.17) under (4.3), (4.4) and (4.18) is straightforward. It is essentially the same as the pure gravitational part of BD theory, which invariance under (4.3), (4.4) and (4.18) (with $\alpha = \omega + \frac{3}{2}$) has been checked, for instance, in [10]. For the matter part of (3.16) (and correspondingly (4.17)) we have that, under (4.3)

$$S_{GR}^{matter} = 16\pi\int d^4x\sqrt{-\hat{g}}e^{2\psi}L_{matter} = 16\pi\int d^4x\sqrt{-\bar{g}}e^{2(1-\sigma)\psi}L_{matter}, \tag{4.19}$$

since $\sqrt{-\hat{g}} = e^{-2\sigma\psi}\sqrt{-\bar{g}}$. Hence, taking into account (4.4) we complete the demonstration that (4.17) (and correspondingly (3.16)) is invariant in form under the transformation of units (4.3), together with the scalar field redefinition (4.4) and the free-parameter transformation (4.18). Another evidence of the consistency of GR theory as formulated in the string (or the Jordan) frame is that it is naturally linked with Weyl geometry based on the postulates (4.9) and (4.10). This geometry itself is invariant under (4.3) and (4.4). Hence, we reach to the conclusion that the conformal formulation of general relativity (string frame formulation or Jordan frame one) is a consistent formulation of the laws of gravity. Unfortunatelly this is not true neither for the canonical formulation of general relativity nor for BD theory.

V. WEYL GEOMETRY AND THE QUANTUM

There are some results that hint at Weyl geometry as a geometry that can take account, in a natural way, of the quantum effects of matter. The first, but not the unique, has already been presented in section **IV**. In fact, Weyl

[2]For testing BD theory, transformations (4.3) and (4.4) should be completed with a transformation of the BD coupling constant $\bar{\omega} = \frac{\omega - \frac{3}{2}\sigma(\sigma-2)}{(1-\sigma)^2}$.

geometry is already invariant under units transformations, as required for any geometrical setting where to describe the physics that is itself insensible to the transformations of the units of measure one chooses.

The second is connected with an idea presented in reference [8]. These papers were based on the de Broglie-Bohm quantum theory of motion [7]. The authors showed that the quantum effects of matter, being explicit in one frame through the following expression, $(\nabla S)^2 = m^2(1+Q)$, where S is the canonical action for the matter fields, m is the constant mass of the matter particles and Q is the matter quantum potential, can be hidden in a conformal transformation of the kind (1.1) in the conformal formulation of this expression. They concluded that the quantum effects of matter are already contained in the conformal metric they called as 'physical metric'. They were led to this conclusion since, the non-geodesic motion of matter particles in one frame (due to the quantum force), is mapped into (apparently) a geodesic motion in the conformal frame if one considers that in (1.1) $\Omega^2 = 1+Q$. We are not concerned here with the validity of these results. Our approach is a little different although the leading idea is the same. We shall take the action (3.3) for GR in the Einstein's formulation that does not contain the quantum effects. In this formulation material test particles follow the geodesics of the Riemann geometry

$$\frac{d^2x^a}{ds^2} = -\Gamma^a_{mn}\frac{dx^m}{ds}\frac{dx^n}{ds}. \tag{5.1}$$

Under the conformal rescaling (1.1) with $\Omega^2 = e^{-\psi}$, this equation is mapped into

$$\frac{d^2x^a}{d\hat{s}^2} = -\hat{\Gamma}^a_{mn}\frac{dx^m}{d\hat{s}}\frac{dx^n}{d\hat{s}} - \frac{1}{2}\psi_{,n}\left(\frac{dx^n}{d\hat{s}}\frac{dx^a}{d\hat{s}} - \hat{g}^{an}\right). \tag{5.2}$$

If we set $e^{-\psi} = 1+Q$, i.e., $\psi = -\ln(1+Q)$, where Q is the matter quantum potential, hence we can consider that the second term in the rhs of eq.(5.2) is the quantum force. This means that under the conformal rescaling, the classical motion given by (5.1) is mapped into a quantum motion in the conformal frame. However, we should realize that eq.(5.2) defines, in fact, a time-like geodesic in a Weyl manifold (as discussed in previous sections). Hence, following the leading idea in reference [8], i.e., $\psi = -\ln(1+Q)$, where Q is the matter quantum potential, we can conclude that Weyl geometry contains implicitly the quantum effects of matter, i.e., it is already a quantum geometry. These ideas will be developed in full detail in a forthcoming paper.

The third result that hints at Weyl geometry as a geometry that implicitly contains the quantum effects of matter has been already presented in subsection **A** of section **III**. In the usual Einstein's formulation of general relativity, the occurrence of spacetime singularities is inevitable if the matter obeys some reasonable energy conditions [6]. It is usually linked with the lack of quantum considerations in this formulation of GR. This can be thought of as a property of Riemann spacetimes in general. Hence, we can regard Riemann geometry as an 'incomplete' geometry since it must be 'completed' with the inclusion of quantum effects. In general, we are tempted to call geometries compatible with geodesically incomplete spacetimes as incomplete geometries. Then we are led to consider the incompleteness of a given geometry as due to the fact that it does not consider the quantum effects of matter.

As we have already shown in subsection **A** of section **III** for FRW cosmologies (see references [5,9]), singularities occurring in usual Einstein's GR (action (3.3)) that is naturally linked with Riemann geometry, are removed in the conformal formulation (action (3.16)) that is naturally linked with Weyl geometry. Singular Riemannian spacetimes are mapped under the conformal rescaling (1.1) into wormhole (singularity-free) spacetimes on a Weyl geometry. In the same way, in reference [4], it was shown that a Riemann spacetime with a Schwarzschild black hole is mapped, under a transformation of the kind (1.1), into a wormhole (geodesically complete) spacetime in Weyl geometry. In other words, in these cases the incomplete Riemann geometry is conformal to a complete Weyl geometry, i.e., a geometry that is compatible with geodesically complete spacetimes (for the given range of the free parameter). We hope that completeness of Weyl geometry means that it implicitly contains the quantum effects of matter.

If our considerations here are correct then, we can reach to the following conclusion. Complete Weyl geometries should be taken as a proper framework for describing the physical laws of nature without the unnatural separation of physics in classical and quantum laws. Correspondingly, string frame (or Jordan frame) GR compatible with Weyl geometry, provides an intrinsically quantum description of the physics.

VI. IS THE GEOMETRY OF THE WORLD UNIQUE?

The fact that physical observations can not distinguish a Riemann manifold with singularities from a Weyl manifold without them, is very striking. Experimental observations show, in particular, that there are several astrophysical black holes located in our universe. One of them is located in the center of our own galaxy.

In reference [4] it has been shown that under a conformal rescaling of the kind (1.1), what appears as a Schwarzschild black hole in a Riemann spacetime is mapped into a wormhole (singularity-free) spacetime on Weyl geometry. Since the conformal transformation (1.1) does not touch the spacetime coincidences (coordinates), i.e., spacetime measurements are unchanged under this transformation, then we can conclude that a Riemannian Schwarzschild black hole is observationally indistinguishable from a wormhole in Weyl geometry. Although an astrophysical black hole does not has the high symmetry inherent to a Schwarzschild black hole, we expect that under (1.1) it is transformed into a some kind of astrophysical wormhole or a similar astrophysical object without singularity. Hence we arrive at a kind of 'duality' of the geometrical representation of our real (observationally testable) world. Our previous discussion resolves this 'duality'. In fact, Riemann geometry with the undesirable occurrence of singularities is not a consistent framework for confronting our experimental observations. The laws of Riemann geometry are not invariant under a general transformation of units (in particular the transformation of units studied in section **IV**) while the experimental measurements should not depend on the particular values of the units of measure one chooses. We reach to the conclusion that the occurrence of black holes (in particular astrophysical black holes when experimental observations are concerned) is due to a wrong choice of the geometrical framework for describing the physics. Weyl geometry seems to be this correct geometrical framework since its basic laws (equations (4.9) and (4.10)) are not affected by the units transformation (4.3) and the scalar field redefinition (4.4). Hence, although the occurrence of compact astrophysical objects has been made evident by the experimental observations, these should not be confronted as black holes containing singularities but as wormholes without them.

Although the geometrical 'duality' discussed in reference [4] can be resolved with the help of considerations like those just given in this section, a source of ambiguity remains in the geometrical interpretation of the physical reality. This ambiguity is linked with units transformations of the kind (4.3), (4.4) and is, in principle, unavoidable. In fact, as shown in section **IV**, under the transformations (4.3) and (4.4), the basic laws of Weyl geometry are invariant in form (as required). This means that there exist different metrics

$$g^{\sigma}_{ab} = e^{\sigma\psi}\hat{g}_{ab}, \tag{6.1}$$

with arbitrary $\sigma \neq 1$, that are uniquely linked with Weyl geometries with different functional laws of change of the scalar product (see eq.(4.10))

$$dg^{\sigma}(\mathbf{Y},\mathbf{Y}) = (\sigma-1)dx^n\psi_{,n}g^{\sigma}(\mathbf{Y},\mathbf{Y}), \tag{6.2}$$

i.e., $g^{\sigma}(\mathbf{Y},\mathbf{Y}) \sim e^{(\sigma-1)\int dx^n\psi_{,n}}$.

All of these conformal Weyl geometries are equally consistent frameworks for the description of the physics and are experimentally indistinguishable.

We are then led to a kind of postulate of equivalence of geometries: There exists an infinite set of spacetimes $(\mathcal{M},\mathbf{g}^{\sigma})$ uniquely linked with an infinite set of Weyl geometries with different laws of length transport given by (6.2) ($\mathcal{M}$ is a smooth Weyl manifold), that are physically equivalent and equally consistent, for the description of the physical laws. This can be viewed as an extension of the postulate of equivalence of coordinate systems linked by general coordinate transformations in GR.

The consequence of the postulate of equivalence of conformal Weyl geometries for the description of the physical reality we shall call as geometrical relativity. We think this will represent a natural extension of general relativity to include invariance under the one-parameter Abelian group of units transformations and, if our considerations in section **V** are correct, to implicitly include the quantum effects of matter.

Finally we shall remark that the relativity of geometry implies nothing more than a relativity of the geometrical interpretation of the physical reality. Physical reality itself is unique.

It will be of interest, in the future, to look at more general transformations of units than those given by (4.3) and (4.4), in order to further extend our results.

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